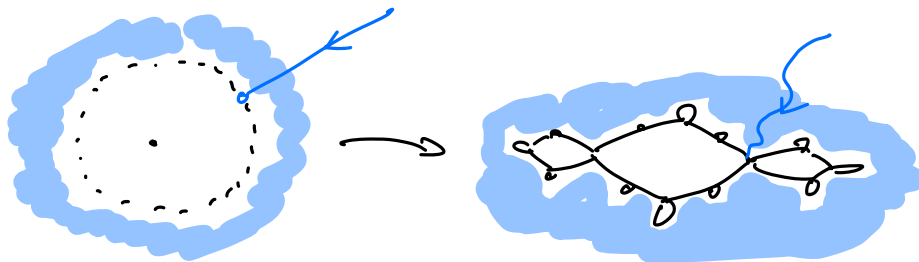


Local Connectivity

$f \in \mathbb{C}[z]$; $J(f)$ is connected $\Leftrightarrow \text{Crit}(f) \subset K$

$$\gamma: \hat{\mathbb{C}} \setminus \bar{D} \xrightarrow{\sim} \hat{\mathbb{C}} \setminus K$$



External rays

$$R_\theta = \{ \gamma(re^{2\pi i \theta}) \mid r > 1 \}$$

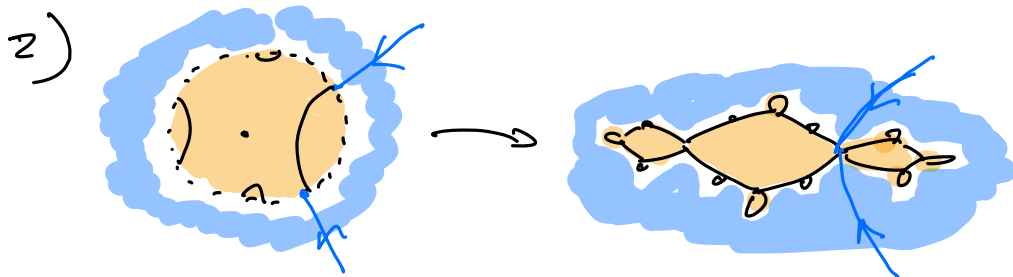
level at $z \in J$ if
 $\lim_{r \rightarrow 1^+} \gamma(re^{i\theta}) = z$.

Q: Does γ extend continuously to a map

$$\hat{\mathbb{C}} \setminus D \rightarrow \overline{\hat{\mathbb{C}} \setminus K}?$$

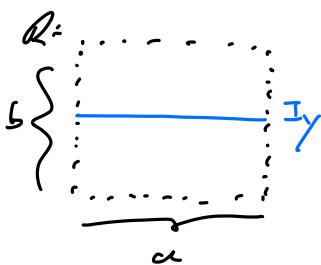
1) All external rays level, and we get

$\gamma: S^1 \rightarrow 2K = J$, $J \cong S^1 / \sim_\gamma$ since γ is
 a closed map because J is compact



Collapse along
 $\theta_1, \theta_2 \in S^1$ iff
 $\gamma(\theta_1) = \gamma(\theta_2)$

Pinched disk model $K(f) \cong D / \sim$



$$R = (0, a) \times (0, b)$$

Lemma (Length-Area inequality)

Let $\gamma: R \rightarrow \mathbb{C}$ be a lab map so the

area $A = \text{area}(\phi(R)) < \infty$. Then for a.e.
 $y \in (0, b)$, $\text{length}(\phi(I_y)) < \infty$. Moreover,
 there exists a y st.

$$\text{length}(\phi(I_y)) \leq \left(\frac{a}{b} \text{Area}(\phi(R)) \right)^{1/2}.$$

Proof. $L(y) := \text{length}(\phi(I_y))$
 $= \int_0^a |\phi'(x+iy)| dx$

$A = \text{Area}(\phi(R)) = \int_R |\phi'(x+iy)|^2 dx dy$

$L(y)^2 \leq a \cdot \left(\int_0^a |\phi'(x+iy)|^2 dx \right)$
 (C-S)

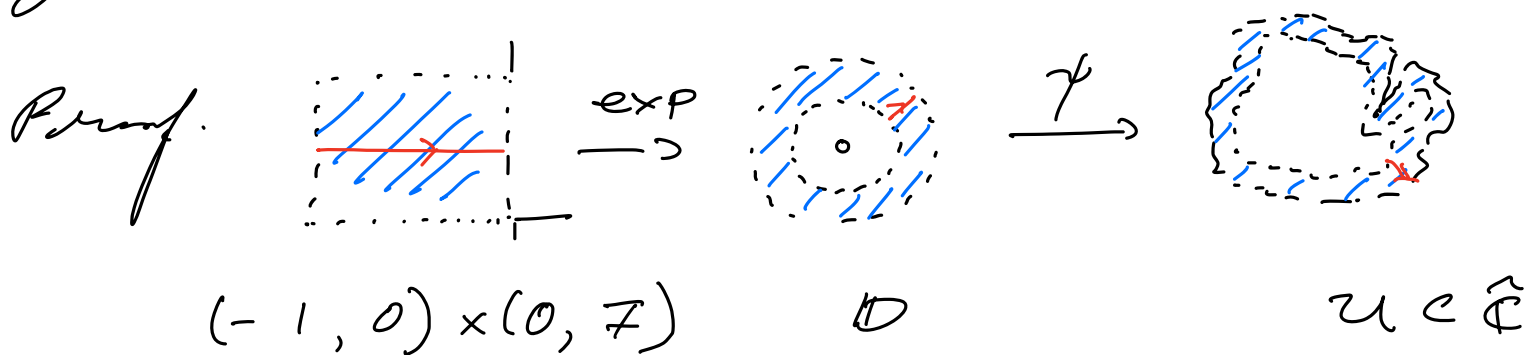
Change of $L(y)^2$:

$$\frac{1}{b} \int_0^b L(y)^2 dy \leq \frac{a}{b} \int_R |\phi'(x+iy)|^2 dx dy = \frac{a}{b} A$$

For some y , $L(y) \leq \sqrt{\frac{a}{b} A}$, and for a.e., finite.

Thm. (Fatou) Let $\psi: \mathbb{D} \rightarrow \mathbb{C}$. Then for
 a.e. θ , the map $r \mapsto \psi(re^{i\theta})$ has a
 limit. For any $z_0 \in \partial U$, the set of
 angles of rays that land at z_0 has measure

zero.



Under $\gamma \circ \exp$, a.e. I_γ has finite length in the image and hence a well-defined limit as $r \rightarrow 1^-$.

Thm. (F. and M. Riesz) Let $f: \mathbb{D} \rightarrow \mathbb{D}$ be hol.

For any $c_0 \in \overline{\mathbb{D}}$, $\{t \mid \lim_{r \rightarrow 1^-} f(re^{it}) = c_0\}$

has measure zero.

$\rightarrow$ If in $\mathcal{U} \subset \hat{\mathbb{C}}$ is $\cong \mathbb{D}$, $\hat{\mathbb{C}} \setminus \mathcal{U} \supset \{0, \infty\}$ w.l.o.g.

Suppose $e^{i\alpha} [0, \infty]$ avoid c_0 . Then $\sqrt{\gamma}$ lifts to

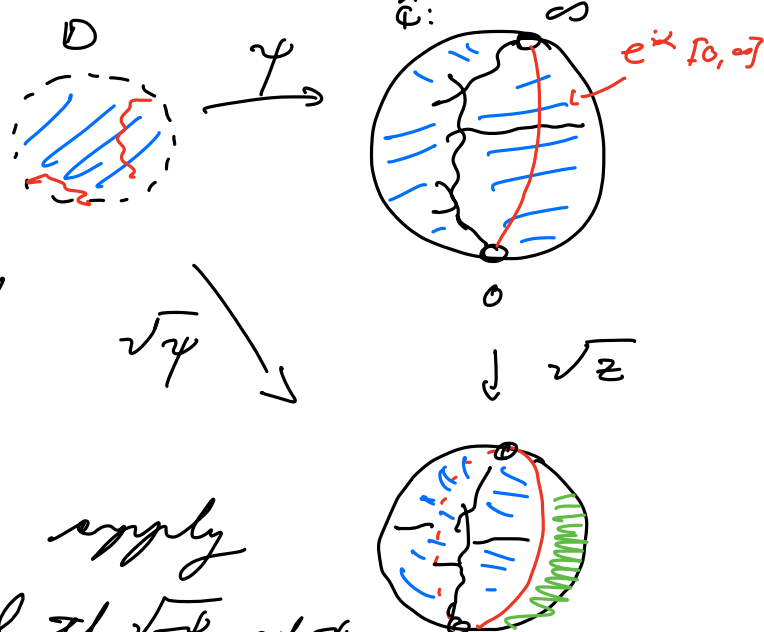
$\hat{\mathbb{C}} \setminus \gamma^{-1}(e^{i\alpha} [0, \infty])$ and

cuts a half-plane.

When it avoids $\hat{\mathbb{C}} \setminus \mathbb{D}$, apply

Riesz + Riesz to conclude for both $\sqrt{\gamma}$ and γ .

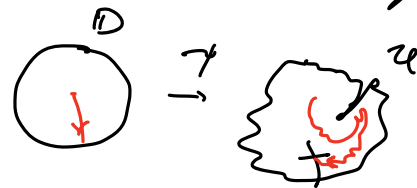
Def. X Hausdorff is locally connected if for



each $x \in X$ and open $U \ni x$, $\exists V \subset U$ open, connected, $V \ni x$.

Thm. (Carathéodory - Jordan) $\gamma: \mathbb{D} \xrightarrow{\sim} U \subset \mathbb{C}$ open

The following are equiv:



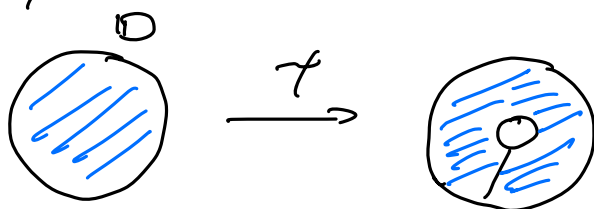
1) ∂U is locally connected

2) $\bar{U}$ is locally connected

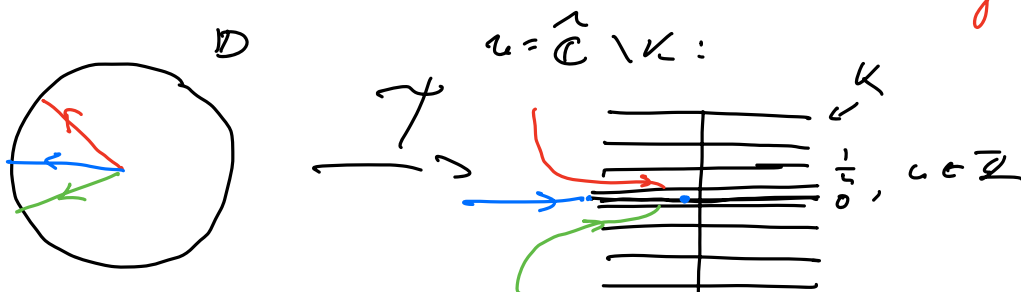
3) γ extends to a continuous map $\bar{\mathbb{D}} \rightarrow \bar{U}$

4) Every map $r \mapsto \gamma(re^{it})$ tends to $\gamma(e^{it})$, where $\gamma: S^1 \rightarrow \partial U$ is continuous.

* $\gamma: \bar{\mathbb{D}} \rightarrow \bar{U}$ usually not a homeo:



* All maps may tend even if ∂U is not locally connected:



$\gamma: S^1 \rightarrow K$ is not cont. on S^1

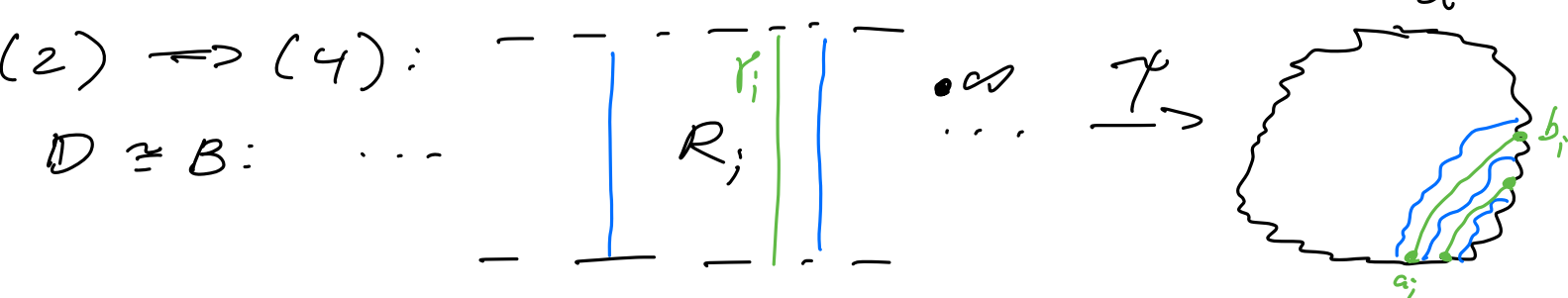
Proof. (1) $\Leftrightarrow$ (2) HW, (3) $\Leftrightarrow$ (4)

(4) $\Rightarrow$ (1): $\gamma \in \partial U$, $\gamma(x) = \gamma$. Pick $x_n, x'_n \in S^1$

s.t. $x \in (x'_n, x_n)$, $x_n, x'_n \rightarrow x$. By



inf cond, diam $\varphi(x_{u'}, x_u) \rightarrow 0$
 and each $(x_{u'}, x_u)$ is connected & open.



- $\text{Area}(\varphi(R_i)) \rightarrow 0$, so $\exists$ mult. $\gamma_i \subset R_i$ s.t.
 $\text{length}(\varphi(\gamma_i)) \rightarrow 0$, $\overline{\varphi(\gamma_i)} \cap \partial U = \{a_i, b_i\}$.
- $\varphi(\gamma_i)$ are cross-cuts: " $\gamma \subset U$ home $\cong (0,1)$,
²⁾ $\overline{\gamma}$ home $\cong [0,1]$, ³⁾ $\overline{\gamma} \cap \partial U = 2$ point.

Lemma. A cross-cut divides U into two components.
 Proof. $\overline{U} \xrightarrow{\gamma} \overline{U}/\partial U \cong S^2$, Jordan curve thm.
 $\overline{\gamma} \xrightarrow{\gamma} S^1$

Let N_i be one of the components.

Not true for:

Lemma. (Short elem, small lake)

The diagram shows a lake (region) with several horizontal lines crossing it, representing short elements. A blue arrow points from the text to the diagram.

$\forall \varepsilon > 0 \exists \delta > 0$ s.t. if γ_i is a crosscut of diam $< \delta$,
 then diam $N_i < \varepsilon$.

Proof. 1) Local connectivity $\Rightarrow \forall \varepsilon' > 0 \exists \delta' > 0$ s.t. if
 $d(a, b) < \delta'$, then $\exists K \subset \partial U$ connected, diam $(K) < \varepsilon$,
 $K \ni a, b$.

2) $\overline{\gamma} \cap \partial U = \{a, b\}$, $d(a, b) < \delta'$,
 $\{a, b\} \subset K$ diam $(K) < \varepsilon'$ K connected.



3) Claim. $K \cup \gamma$ separates N from $U \setminus \bar{N}$:

Pick $x \in N$. If $K \cup \gamma$ did not separate,



$\exists$ arc A from $x \in N$ to $y \in U \setminus \bar{N}$ disjoint from $K \cup \gamma$.

Pick arc A' from x to y cutting through γ once. Together $A \cup A'$ would separate a, b from each other, disconnect K .

4) $\text{diam } N \leq \text{diam}(K \cup \gamma) \leq \varepsilon' + \delta'$. Pick $\varepsilon < \varepsilon'$.

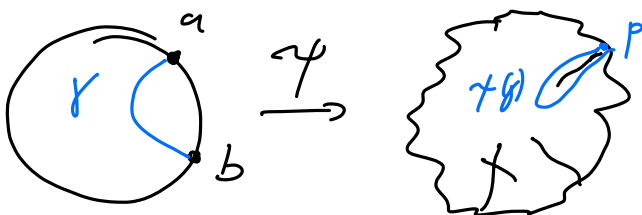
• As $\text{diam } N_i \rightarrow 0$, γ extends continually to ∂D .

• $\gamma(\bar{D}) \subset \bar{U}$ is closed and contains U , so $\gamma(\bar{D}) = \bar{U}$.

Cor. $\gamma: \bar{D} \rightarrow \bar{U}$ is a homeo iff ∂U is a Jordan curve.

Proof. $\rightarrow$ by def.

$\Leftarrow$: surjectivity because if $\gamma(a) = \gamma(b) = p$ then we can connect a to b via $\gamma \subset \bar{D}$. Then $\gamma(\gamma)$ disconnects ∂U , but a



Jordan curve can't be disconnected by removing one point.

- HCC compact, connected, $\hat{\mathbb{C}} \setminus H$ simply-connected
- $\gamma: \hat{\mathbb{C}} \setminus \overline{D} \rightarrow \hat{\mathbb{C}} \setminus H$

1) Define $\theta_1 \sim \theta_2$ if $\theta_1 \neq \theta_2$ and

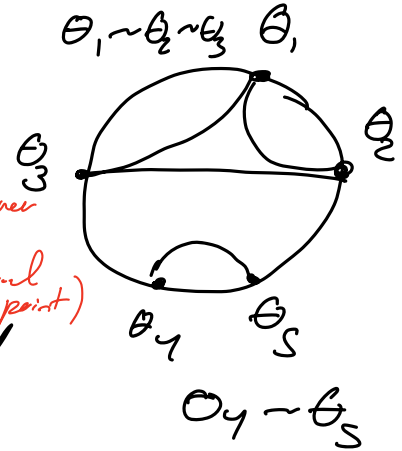
$$\lim_{r \rightarrow 1^+} \gamma(r e^{2\pi i \theta_1}) = \lim_{r \rightarrow 1^+} \gamma(r e^{2\pi i \theta_2}).$$

(H loc. connected $\Rightarrow 2H \cong S^1 / \sim$)

2) If $\theta_1 \sim \theta_2$, then connect them in the geodesic in D .

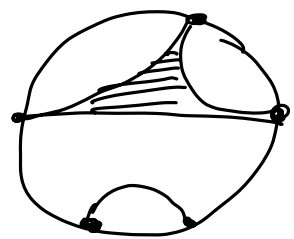
Def. Geodesic lamination is a closed subset of D that is a disjoint union of geodesics.

(We need $\theta_1 \sim \theta_2$ to be an equivalence relation! Minor technical point)



3) If we take the closure of $\sim \Rightarrow$ lamination.

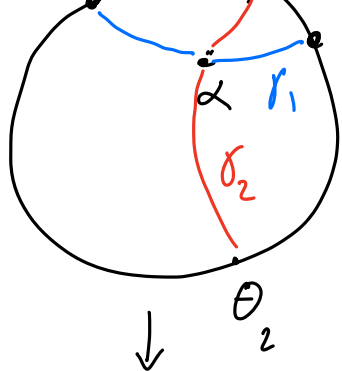
4) Extend $\sim$ to D by taking convex hulls of each equiv. class.



Lemma. The convex hulls are disjoint.

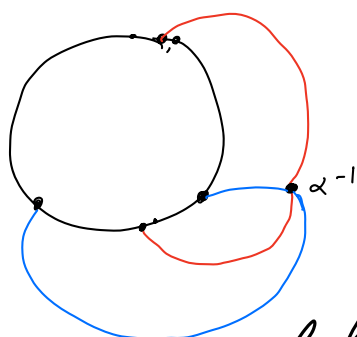
Proof. α lies in both hulls H_1, H_2 . Connect α to points $\theta_1, \theta_1' \in H_1, \theta_2, \theta_2' \in H_2$ on the





boundary map γ is given
 paths $\gamma_1 \subset H_1, \gamma_2 \subset H_2$. The map $z \mapsto \frac{1}{z}$ and map under ψ :

But then the



→

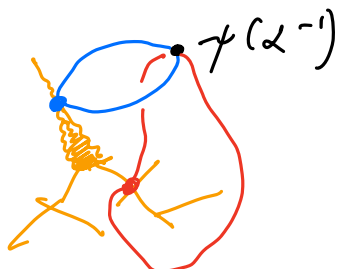


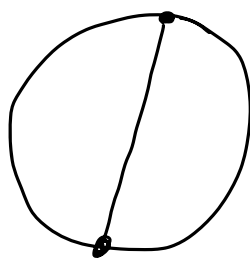
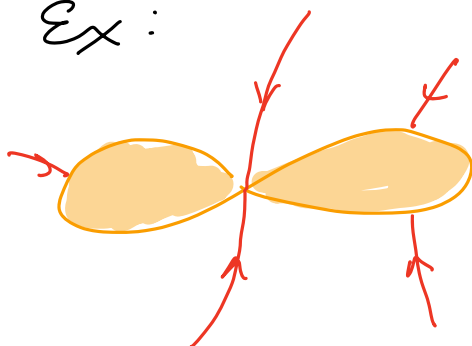
image of γ_1, γ_2 will
 only intersect at
 $\psi(\alpha^{-1})$ transversely.

5) Take the closure of the classes again.
Colored grooved laminator : Z is the gaps
 $D \setminus Z$ colored black or white so no two are
 adjacent.

Colored laminator $\iff$ Equi skeleton

6) Take the gap by Z :

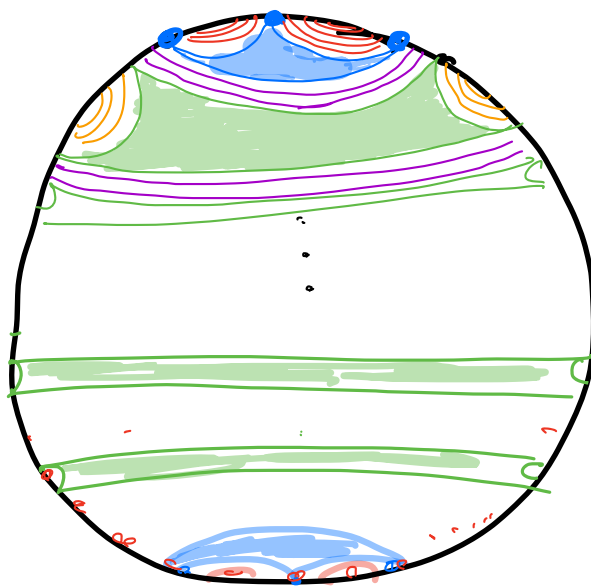
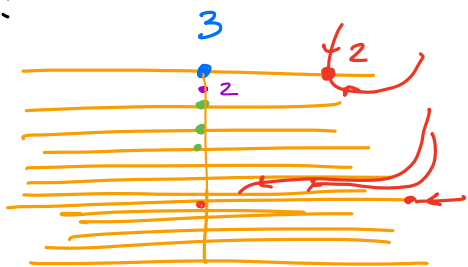
Ex:



All other are
 right

Pinching

Ex:



Ex: Show Schleicher pictures

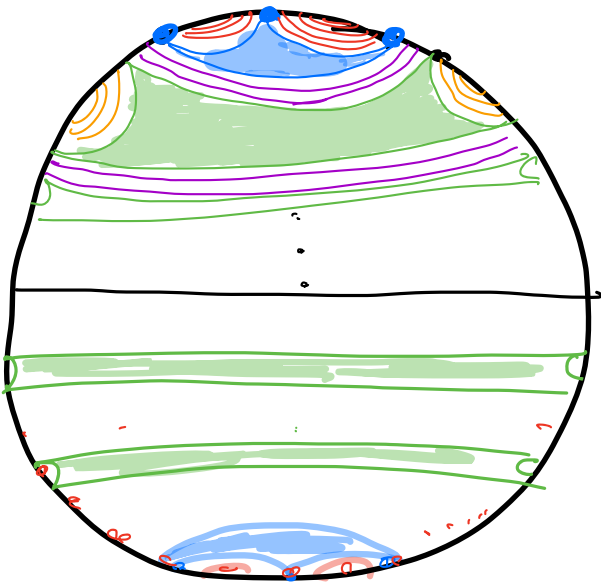
Prop. (Locally Connected Model) Suppose that the classes are dense in S' . Then $\bar{\mathbb{D}}/\mathbb{Z}$ is locally connected, $\hat{\mathbb{C}} \setminus (\bar{\mathbb{D}}/\mathbb{Z})$ is simply-connected, and the map $\phi = \gamma^{-1}: \hat{\mathbb{C}} \setminus H \xrightarrow{\sim} \hat{\mathbb{C}} \setminus \bar{\mathbb{D}}$ extends to a cont. singulator

$$\phi: (\hat{\mathbb{C}}, H) \longrightarrow (\hat{\mathbb{C}} \cong \hat{\mathbb{C}}/\mathbb{Z}, H/\mathbb{Z}).$$

• This is a homeo iff H is locally connected.

Ex: Schleicher cycles

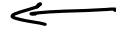
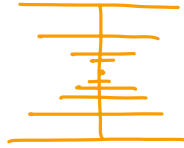
Ex: For each, take the closure of the laminations.



LC model:

H/Z :

∴



H:

